

Diagram Effective or Diagram Dependent?

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This paper will focus on two students who depended on diagrammatic representations in both a Fraction Screening Test and in a subsequent Structured Interview. One student attempted to use diagrams, with limited success, to identify the correct relationships, and consequently struggled to generalise her strategies as she responded to the questions presented in the interview. The other student used diagrams more effectively, and was able to move from a reliance on diagrams to using a partially multiplicative solution strategy. When supported by strong number knowledge, diagrams are an effective means for helping to solve reverse fraction tasks but may hinder students attempts to generalise their thinking.

The links between fractional knowledge and readiness for algebra have been highlighted by many researchers such as Wu (2001); Jacobs, Franke, Carpenter, Levi, and Battey, (2007); and Empson, Levi, and Carpenter, (2011). Siegler et al. (2012) used longitudinal data from both the United States and United Kingdom to show that competence with fractions and division in fifth or sixth grade is a uniquely accurate predictor of students' attainment in algebra and overall mathematics performance five or six years later. This paper focuses on the final stage of an Australian research study that investigated the links between fractional competence and algebraic thinking. For our research, emergent algebraic thinking is defined in terms of students' capacity to identify an equivalence relationship between a given collection of objects and the fraction this collection represents of an unknown whole, and then to operate multiplicatively on both to find the whole. We also anticipated that some students would be able to generalise their solutions, providing even more convincing evidence of algebraic thinking.

Researchers have advocated the use of diagrams as a problem-solving strategy for students solving unfamiliar problems. Diezmann and English, (2001) stated that a diagram is a visual representation that presents information in a spatial layout. The appropriateness of a diagram for the solution of a problem depends on how well it represents that problem's structure. Booth and Thomas (2000) suggested that while diagrams are useful for some students, other students may not see the structure of the problem in diagrams or may be unfamiliar with the use of diagrams in the problem-solving process.

This paper will focus on two students, each of whom initially appeared to depend on diagrammatic representations when solving the Structured Interview tasks. We address the research question: *What aspects of the use of diagrams helps or hinders the development of emergent algebraic thinking?*

The Study

In this research middle years' students completed two paper and pencil tests: the Fraction Screening Test and an Algebraic Thinking Questionnaire (Pearn & Stephens, 2015). Later a Structured Interview was used with 45 students from two schools, 19 Year 5 and 6 (10-12 years

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old) students and 26 Year 8 (14 years old) students. One of the students reported in this paper was in Year 5, the other in Year 8. Responses across the Structured Interview tasks revealed that while some students struggled to move on from the additive strategies they used in paper and pencil tests, others used more robust generalisations (Pearn & Stephens, 2017).

The three reverse fraction tasks from the Fraction Screening Test (Pearn & Stephens, 2015; 2017) provided an initial lens into the different types of students' strategies. These are called *reverse fraction tasks* as students need to find the number of objects representing the whole when given the number of objects representing a given fractional part.



Reverse Fraction Task 1	Reverse Fraction Task 2	Reverse Fraction Task 3
<p>This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.</p>  <p>How many counters did I start with? Explain how you decided your answer is correct.</p>	<p>Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs.</p> <p>How many CDs does Kay have?</p> <p>_____</p> <p>Show all your working.</p>	<p>This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.</p>  <p>How many counters did I start with? Explain how you decided that your answer is correct.</p>

Figure 1. The three reverse fraction tasks from the Fraction Screening Test.

Students were chosen to be interviewed only if they had successfully solved at least two of the three reverse fraction tasks. The Structured Interview was designed to investigate whether students who had relied on the use of diagrams or a mix of additive and multiplicative strategies, could because of carefully chosen questions, adopt more consistent multiplicative and generalisable strategies, that are precursors to algebra.

The Structured Interview, included reverse fraction tasks similar to those in the Fraction Screening Test but with progressive levels of abstraction, starting from particular instances and becoming progressively more generalised. The first three questions of the Structured Interview are shown in Figure 2, using the same three fractions as before, without diagrams and with different quantities representing each fraction.

<p>1. Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with.</p> <p>How many counters did I start with?</p> <p>Explain your thinking.</p>	<p>2. Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's.</p> <p>How many CDs does Kay have? _____</p> <p>Explain your thinking.</p>	<p>3. Imagine that I gave you 21 counters which is $\frac{7}{6}$ of the number of counters I started with</p> <p>How many counters did I start with?</p> <p>Explain your thinking.</p>
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Figure 2. Questions 1- 3, Structured Interview

In a second set of three questions (4, 5, and 6), the first part used a new quantity with the same fraction; and the second part started with: "If I gave you *any* number of counters which is also a (given fraction) of the number I started with, what would you need to do to find the number of counters I started with?" Question 4, in Figure 3, is such a question.

<p>4a. If I gave you 18 counters, which is $\frac{2}{3}$ of the number of counters I started with, how would you find the number of counters I started with?</p>	<p>4b. If I gave you <i>any</i> number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with?</p>
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Figure 3. Questions 4a and 4b, Structured Interview.

Students who satisfactorily completed the first six questions of the Structured Interview were asked Question 7 (Figure 4), which required them to use a generalisable method.

What if I gave you *any* number of counters, and they represented *any* fraction of the number of counters I started with, how would you work out the number of counters I started with? Can you tell me what you would do? Please write your explanation in your own words.

Figure 4. Question 7, Structured Interview.

In the Structured Interview, we noted whether students who had relied on additive or subtractive methods, with or without a diagram, used multiplicative methods once the diagrams were no longer provided. We were interested to see whether these deliberately graduated interview questions prompted students to adopt more generalisable methods

Administration of the Interview

The Structured Interview was conducted at each school with four experienced interviewers. At the start of the interview students were shown a copy of their responses to the three paper and pencil reverse fraction tasks. This was then left on the table for students to refer to, if required. The record of interview consisted of interviewers' notes and a three-page document which included the questions and space for students to record their answers and explain their thinking. Students were encouraged to think about, and articulate, their response before writing anything on paper. Students unable to answer Questions 4b, 5b, or 6b were not given Question 7. Each interview took approximately 15 minutes. Students were free to correct their written responses or to exit the interview at any point.

The students' solution strategies for each Structured Interview question were classified using five categories established using the process of the thematic analysis approach suggested by Braun & Clarke (2006). *Diagram dependent* strategies include the use of explicit partitioning of diagrams before using additive or subtractive strategies. *Additive/Subtractive* strategies include those where the student has used addition or subtraction without explicit partitioning of a diagram. Students find the number of objects needed to represent the unit fraction and then use counting or repeated addition to find the number of objects needed to represent the whole. Students using a *partially multiplicative* strategy use both multiplicative and additive methods, by calculating the missing fractional part and then adding it onto the original quantity. Students using *fully multiplicative* strategies find the quantity represented by the unit fraction using division and then multiply that quantity of the unit fraction to find the whole. Students using *advanced multiplicative* methods use appropriate algebraic notation to find the whole, or a one-step method to find the whole by, for example, dividing the given quantity by the known fraction.

Results: Two Case Studies of Gloria and Violet

Reverse Fraction Tasks

In her written response to Reverse Fraction Task 1, Gloria (Figure 5, left), a Year 5 student, used an *additive strategy* to mentally add on another five circles to correctly determine the whole collection was 15. For the same task Violet, Year 8, used a *fully multiplicative* method as shown in the right-hand side of Figure 5. She circled five of the dots given in the diagram and wrote the symbol for one-half above the circled dots. While her written explanation was

brief it demonstrated that she knew that one-third was represented by five dots and she multiplied five by three to find three-thirds or one-whole.



<p>5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.</p>  <p>15</p> <p>a. How many counters did I start with? b. Explain how you decided that your answer is correct.</p> <p>I added 5 more circles and then see if it split evenly/ and it did</p>	<p>5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.</p>  <p>15</p> <p>a. How many counters did I start with? b. Explain how you decided that your answer is correct.</p> <p>$\frac{1}{3}$ is 5 so 5 5×3 is 15</p>
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Figure 5. Gloria's and Violet's responses to Reverse Fraction Task 1.

In Reverse Fraction Task 2 both students used a *partially multiplicative* solution strategy. Gloria (left) wrote her solution in words as shown in Figure 6. Violet (right) drew her own diagrams to solve this task. She initially drew four groups of three circles to represent four-sevenths then drew another three groups of three circles to represent the extra three-sevenths needed to represent seven-sevenths or the whole.

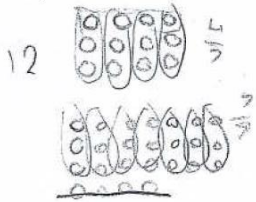
<p>6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? <u>21</u> Show all your working.</p> <p>I knew that 3 goes into 12 4 times and then I added another 9</p>	<p>6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? <u>21</u> Show all your working.</p> 
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Figure 6. Gloria's solution to Reverse Fraction Task 2.

For Reverse Fraction Task 3 Gloria (left) successfully used a *partially multiplicative* strategy and calculated that one-seventh was represented by two counters which she subtracted from the 14 to get 12 as shown in Figure 7. Violet (right) had several attempts at circling sets of dots in her attempt to solve Reverse Fraction Task 3. Her written explanation appears to indicate she is using a *multiplicative strategy* but there is an element of uncertainty in her response as she has written: "started with 12?"



<p>7. This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.</p>  <p>12</p> <p>a. How many counters did I start with? b. Explain how you decided that your answer is correct.</p> <p>well 7 cant go into 14 so I just took away 1 7th of the dots</p>	<p>7. This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.</p>  <p>12 2</p> <p>a. How many counters did I start with? b. Explain how you decided that your answer is correct.</p> <p>$12 \div 2 = 6$ $2 \times 6 = 12$ started with 12? $2 \times 7 = 14$</p>
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Figure 7. Gloria and Violet's responses to Reverse Fraction Task 3.

Gloria consistently used a *partially multiplicative* strategy for each of the three reverse fraction tasks. Violet used a variety of strategies but needed diagrams for all three tasks. The Structured Interview provided additional opportunities to explore the robustness and limitations of the methods used by these two students in the Fraction Screening Test.

Structured Interview

Gloria successfully solved Question 1 (Figure 2) and gave a *partially multiplicative* response as she halved the number representing two-thirds to find the number of counters representing one-third and finally added both amounts together to get three-thirds or the whole. For the same question Violet used a *diagrammatic* approach. She drew three rows of six circles then drew around two of those rows to indicate two-thirds. She correctly wrote that the initial number was 18 counters.

In Question 2 Gloria initially drew eight circles (left-hand side of Figure 8) and initially tried to place the eight circles into seven equal groups. She then reread the question and drew the eight circles in four groups of two and circled each pair (left-hand side of Figure 8) before adding three more pairs of circles. While she did not write that the total was 14 she stated verbally that the answer was 14 CDs.

Violet's initial attempt at using a diagram in Question 2 is incorrect (shown on the right-hand side of Figure 8). She initially drew an array of five rows of four circles then crossed out one circle from each row. She then added two more rows of three circles to make seven rows of three circles. Violet then drew an additional array of seven rows of three circles before attempting to draw around groups of seven circles. She drew around three groups of seven circles, four groups of eight circles and one group of five circles. These attempts are evidence Violet's struggle with multiplication facts. At this stage, she was encouraged by the interviewer to re-read the question. Violet then correctly drew a row of eight circles to represent the four-sevenths, and divided these eight circles into four equal groups, writing the fraction four-sevenths beside the drawing. She then added a further four groups of two circles underneath the first diagram, crossed out one group of two circles, leaving three groups with two circles to represent three-sevenths and correctly stated that the total is 14 CDs. This *partially multiplicative* method is consistent with her solution to Reverse Fraction 2 shown in Figure 6.

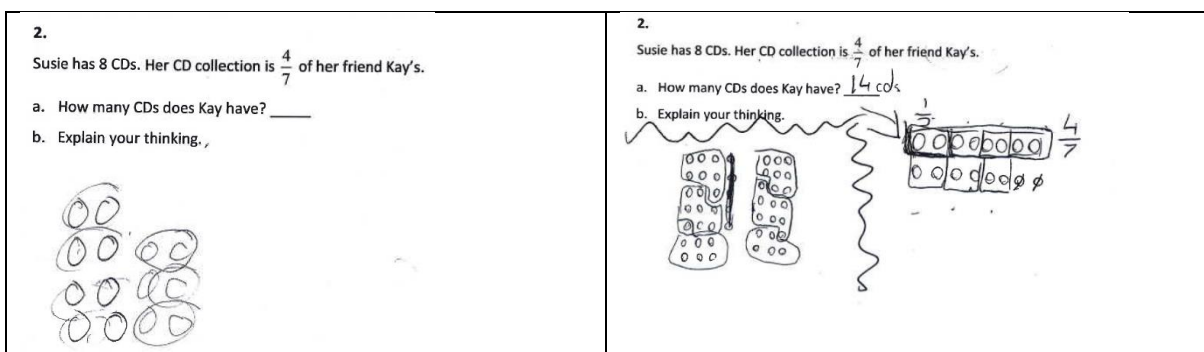


Figure 8. Gloria and Violet's responses to Question 2 of the Structured Interview.

While Gloria was unable to give a correct response for Question 3 Violet correctly partitioned the 21 dots into seven groups of three, recognised that three dots represented one-sixth and correctly stated that there were 18 counters in the whole group. Gloria confidently

answered Question 4 of the Structured Interview using a *partially multiplicative* solution strategy and said: “Half of 18 is 9 so if I add the nine to 18 I get 27 counters. When asked what she would do if she was given ‘any number of counters which was $\frac{2}{3}$ of the number’, Gloria confidently responded: “You would halve the number and then add it to that number”.

As shown in Figure 9, Violet constructed four diagrams but the first, second and fourth are incorrect. In the first diagram, she draws three rows of six circles and draws around each row stating that each row represents one-third. In the second diagram she again draws three rows of six circles and attempts to divide these in two parts but unfortunately, she ends up with one group of seven dots and one of 11 dots. The fourth diagram shows three rows of seven dots, which she divided into three groups of seven, then wrote the answer as 14, which is two of the groups of seven or two-thirds of the 21 dots she drew. The third diagram shows three groups of nine, but only after many corrections have been made. She then correctly decided that three-thirds was 27. After finally succeeding with Question 4a using the third diagram in Figure 9, Violet correctly uses an additive solution for Question 4b which asked about ‘any number of counters’ representing two-thirds saying: “You would half the number and then add the result to the number you started with”.

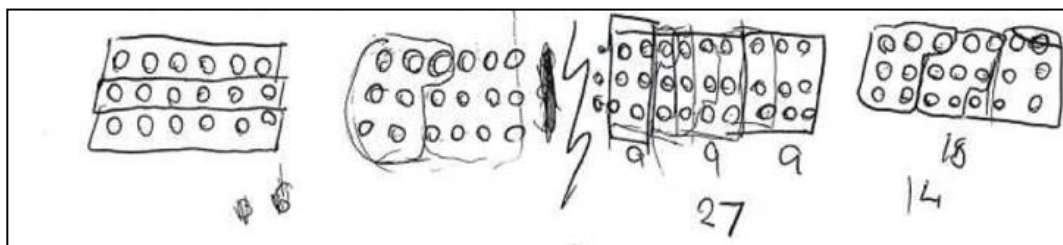


Figure 9. Violet's responses to Question 4 of the Structured Interview.

Gloria used a *partially multiplicative* strategy for Question 5 (see Figure 11). She drew five rows of four circles, then drew vertical lines highlighting the columns of five to show one-seventh of the whole group. She then verbally added on three more groups of five (15) to the original 20 to get 35 CDs. She explained and then wrote, that to calculate the number of CDs needed in the general case of any number of CDs: “whatever number that you have you have to put it into 4 groups and then add another 3 of the groups”.

In Question 5, Violet (right-hand side of Figure 10) correctly finds one quarter of 20 by halving, and halving again, but represents this as three equivalent expressions. She started to draw a diagram which she then scribbled out before using a *partially multiplicative* method to correctly determine three-sevenths as 15 (5×3) and then calculate the whole by adding three-sevenths to the original four-sevenths, which she wrote as $15 + 20 = 35$.

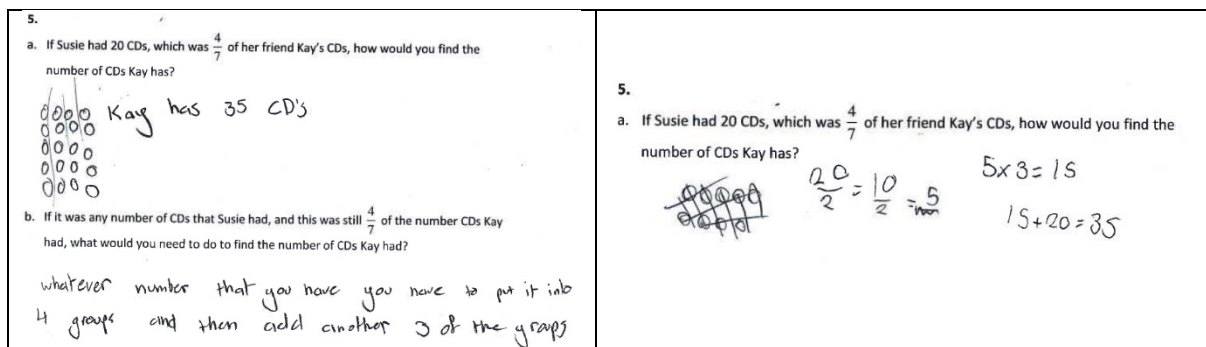


Figure 10. Gloria and Violet's responses to Question 5, Structured Interview.

After completing Question 5, Violet was unable to complete any further questions and the interview was discontinued. Gloria continued with Questions 6 and 7, successfully responding to Question 6a by drawing seven circles to represent the 70 counters and stated that there would be seven groups with 10 counters in each group. To find the number of counters in the whole group she said that she would need to remove one group of ten to get the answer 60. For the general case of ‘any number of counters’ representing the fraction seven-sixths in Question 6b she stated: “Put it into 7 groups. However, many in that group take it away from the original number”. While this demonstrates her use of the *partially multiplicative* strategy as she calculates one-sixth and subtracts that number of counters away from seven-sixths to find six-sixths. Her successful *subtractive strategy* still appears to rely on a diagram to assist in using this method.

Gloria’s response for Question 7 involving ‘any fraction’ with ‘any number of counters representing that fraction’ showed that she used the same *partially multiplicative* strategy she had used for the previous tasks when she stated: “Whatever the numerator is put it into however many groups you (need) then either add or subtract that number”. While this strategy may work for fractions like two-thirds and seven-sixths, with other fractions it is unclear how many times that number might need to be added or subtracted.

Discussion

Gloria’s use of diagrammatic strategies draws attention to a clearly established pattern of representing a whole as a composite of its fractional parts. The underlying conception is that of part-part-whole. Diagrams, often with circling, are used to identify, usually successfully, the component relationships; recognising that it is necessary to deduce the value of the unit fraction, in order to scale up (or down) the number of fractional parts to make a whole. Apart from the first multiplicative step to create a unit fraction, all other operations are performed additively. When presented with a known fraction representing ‘any number’ Gloria explained how the separate parts or components can be combined to make a whole. But in Question 7 when presented with ‘any fraction’, Gloria’s clearly understood part-part-whole strategies cannot be effectively generalised in the way that a fully multiplicative strategy can be generalised: “Whatever the numerator is, put it into however many groups. You then either add or subtract that number”. However, Gloria’s confident use of part-part-whole strategies gives her a clear advantage over Violet who needed diagrams to aid her attempted calculations when solving the interview questions.

Violet has difficulty in creating an appropriate diagram to represent the number relationships as required by the Structured Interview tasks, often requiring several attempts. She provided a partially multiplicative solution for the partly generalised task where ‘any number’ of counters’ represented two-thirds but could not offer a solution for the generalised version of ‘any number of counters’ for either four-seventh or seven-sixths. As the numbers changed and became bigger for the two-thirds questions Violet’s diagrams became more complex requiring several attempts to partition the numbers.

Conclusion and Implications

As the fractions become less familiar, and the numbers larger, students like Violet who rely on diagrams to partition the numbers, encounter greater difficulty. Proficient multiplicative facts and thinking are needed to partition the numbers to find the appropriate unit fraction and from there to scale up to the whole. When dependence on diagrams is not supported by strong

number knowledge, success becomes limited. Diagrams can be effective when students know the number of objects and the fraction they are representing provided these representations reflect proficient multiplicative thinking. Diagrammatic representations may also help students when thinking about how a given fraction may be scaled up or down to give a whole. However, students who depend on diagrams to scale up or down appear to have difficulty in moving away from part-part-whole *additive or subtractive* strategies. Their diagram dependence seems to prevent them from recognising an underlying multiplicative structure from which a truly generalised solution can be constructed. In this respect, the reliance on diagrams when linked with additive or subtractive strategies may hinder emergent algebraic thinking in the form of a truly generalised solution.

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